Advanced Functions

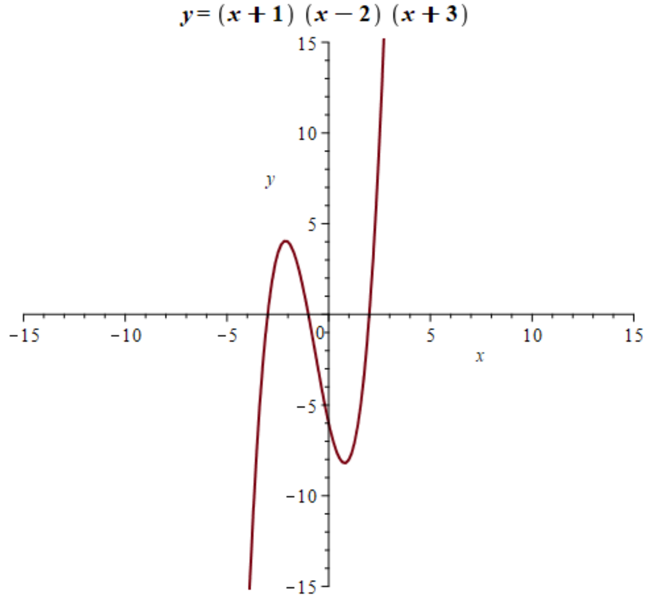
Course Notes

Chapter 2 – Polynomial Functions

***Learning Goals**: We are learning*

- *The algebraic and geometric structure of polynomial functions of degree three and higher*
- *Algebraic techniques for dividing one polynomial by another*
- *Techniques for using division to FACTOR polynomials*
- *To solve problems involving polynomial equations and inequalities*

$y = (x+1)(x-2)(x+3)$



A cubic polynomial

A∞Ω
Math@TD

Chapter 2 – Polynomial Functions

Contents with suggested problems from the Nelson Textbook (Chapter 3)

2.1 Polynomial Functions: An Introduction

Learning Goal: We are learning to identify polynomial functions.

Definition 2.1.1

A **Polynomial Function** is of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots a_2 x^2 + a_1 x^1 + a_0 x^0$$

where n = integers, 0, 1, 2 ...

and a_n are coefficients (any number)

and the exponents are also integers.

Examples of Polynomial Functions

a) $f(x) = 8x^4 - 5x^3 + 2x^2 + 3x - 5$

$a_4 = 8 \quad a_2 = 2 \quad a_0 = -5$

→ constant

b) $g(x) = 7x^6 - 4x^3 + 3x^2 + 2x^1$

$a_6 = 7 \quad a_4 = 0 \quad a_1 = 2 \quad a_0 = 0$

Notes: The **TERM** $a_n x^n$ in any polynomial function (where n is the **highest power** we see) is

called the $\boxed{\text{Leading term}}$, and then we write all the following terms

in $\boxed{\text{descending order.}}$.

The Leading term has two components:

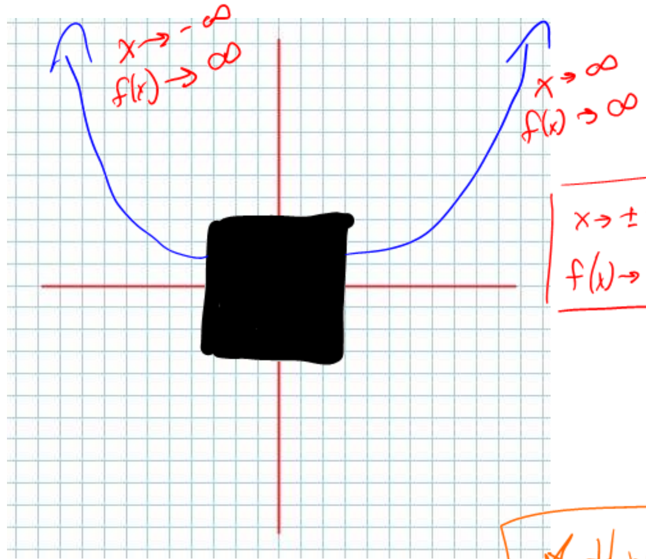
1) Leading coefficient, a_n, is either positive or negative.

2) n, the highest power/degree, it can be even or odd.

The leading term tells us the **end behaviour** of the polynomial function.

☆ all polynomial functions have 4 possible end behaviors.

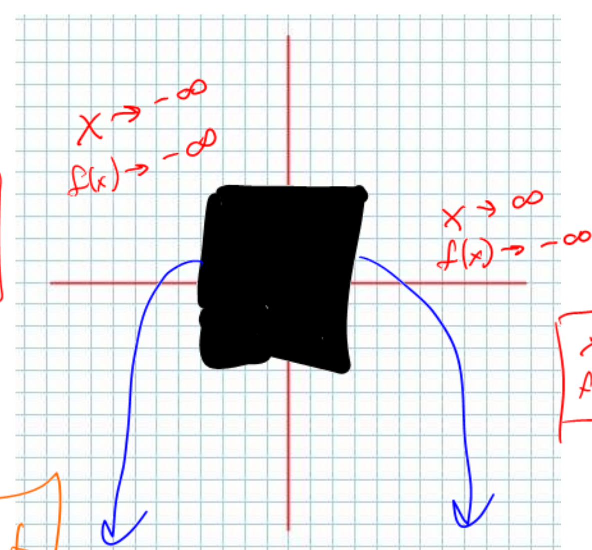
Pictures



$x \to \pm\infty$
$f(x) \to \infty$

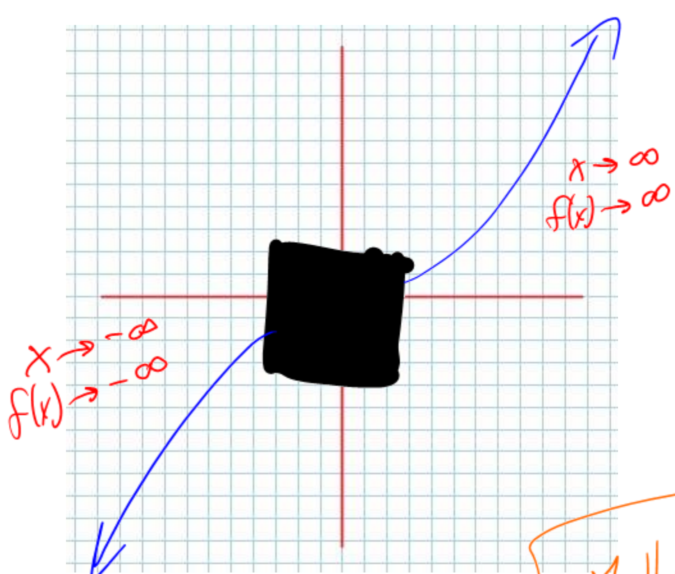
n is even
a_n is positive

☆ think of a parabola



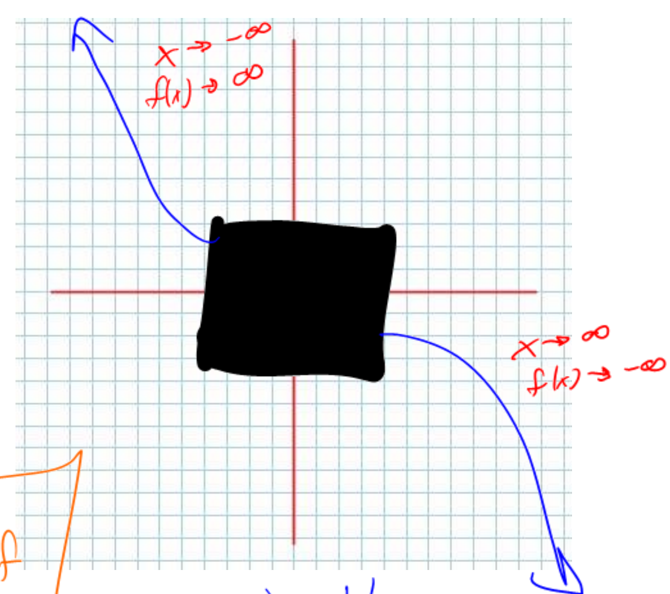
$x \to \pm\infty$
$f(x) \to -\infty$

n is even
a_n is negative



n is odd
a_n is positive

☆ think of a line



n is odd
a_n is negative

Definition 2.1.2

The **order** of a polynomial function is the value of the highest power, or just the degree of the leading term.

same

ex: $g(x) = 2x^3 + 3x^2 - 8x^5 + 1$

The order of $g(x)$ is 5

Determine the end behaviors of:

$h(x) = 2(x-3)^2(2x+8)^3(4x+5)$

All we need is the leading term!

$2(x)^2(2x)^3(4x)$

$= 2(x^2)(8x^3)(4x)$

$= 64x^6$ → even

↳ positive

∴ $x \to -\infty$, $h(x) \to \infty$ $\quad$ $x \to \infty$, $h(x) \to \infty$

Success Criteria:

- I can justify whether a function is polynomial or not
- I can identify the degree of a polynomial function
- I can recognize that the domain of a polynomial is the set of all real numbers
- I can recognize that the range of a polynomial function may be the set of all real numbers, or it may have an upper/lower bound
- I can identify the shape of a polynomial function given its degree

2.2 Characteristics (Behaviours) of Polynomial Functions

Today we open, and look inside the black box of mystery

Learning Goal: We are learning to determine the turning points and end behaviours of polynomial functions.

Consider the sketch of the graph of some function, $f(x)$:

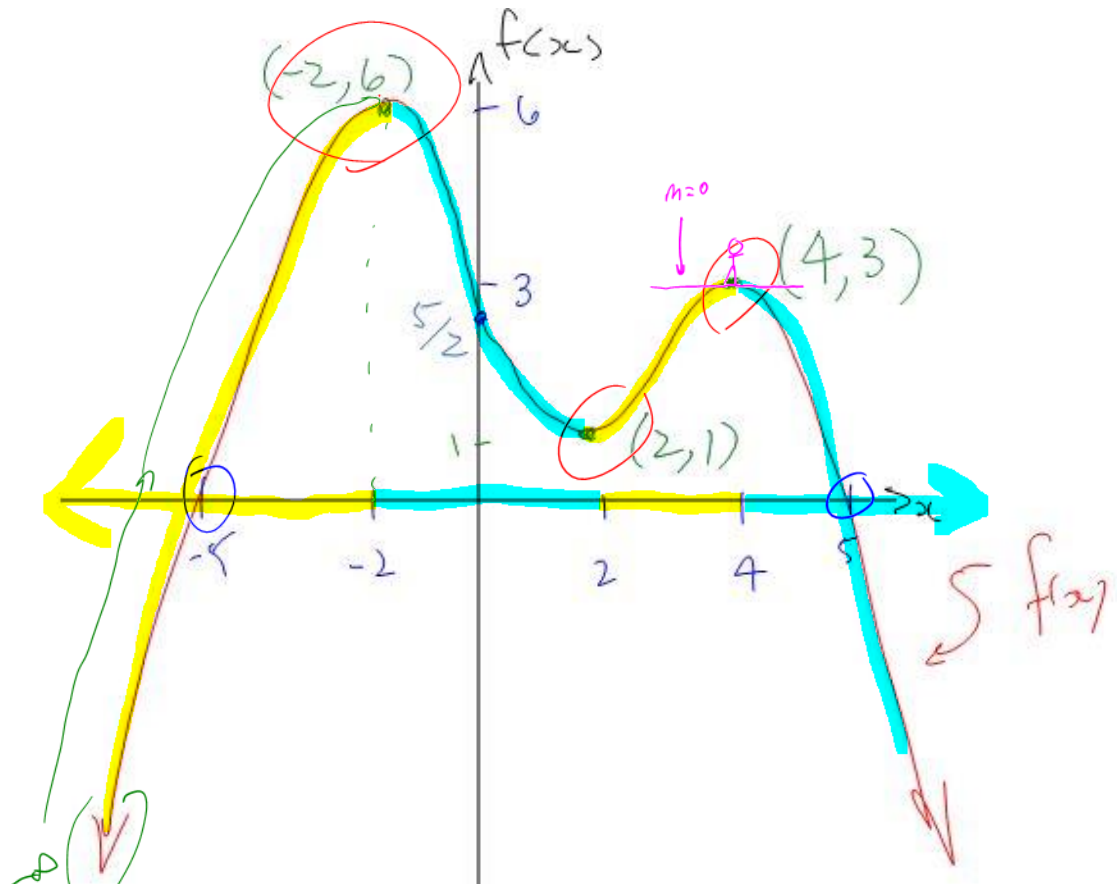


Figure 2.2.1

Observations about $f(x)$:

1) $f(x)$ is a polynomial of even order (degree). The end behaviours are the same.

2) The leading coefficient is negative

3) $f(x)$ has 3 turning points (where the functional behaviour of INCREASING/DECREASING switches from one to the other.)

4) $f(x)$ has 2 zeros (x-intercepts) $f(-5)=0$ and $f(5)=0$

Zeros at $x=-5, 5$

5) $f(x)$ is increasing on $(-\infty,-2)\cup(2,4)$

Only look at x-values

$f(x)$ is decreasing on $(-2,2)\cup(4,\infty)$

6) $f(x)$ has a maximum functional value. of 6.

This max is called the global maximum because it is the absolute highest value.

* only even polynomial functions have a global max/min

7) $f(x)$ has a local minimum at $(2,1)$

and a local maximum at $(4,3)$

Consider the sketch of the graph of some function $g(x)$:

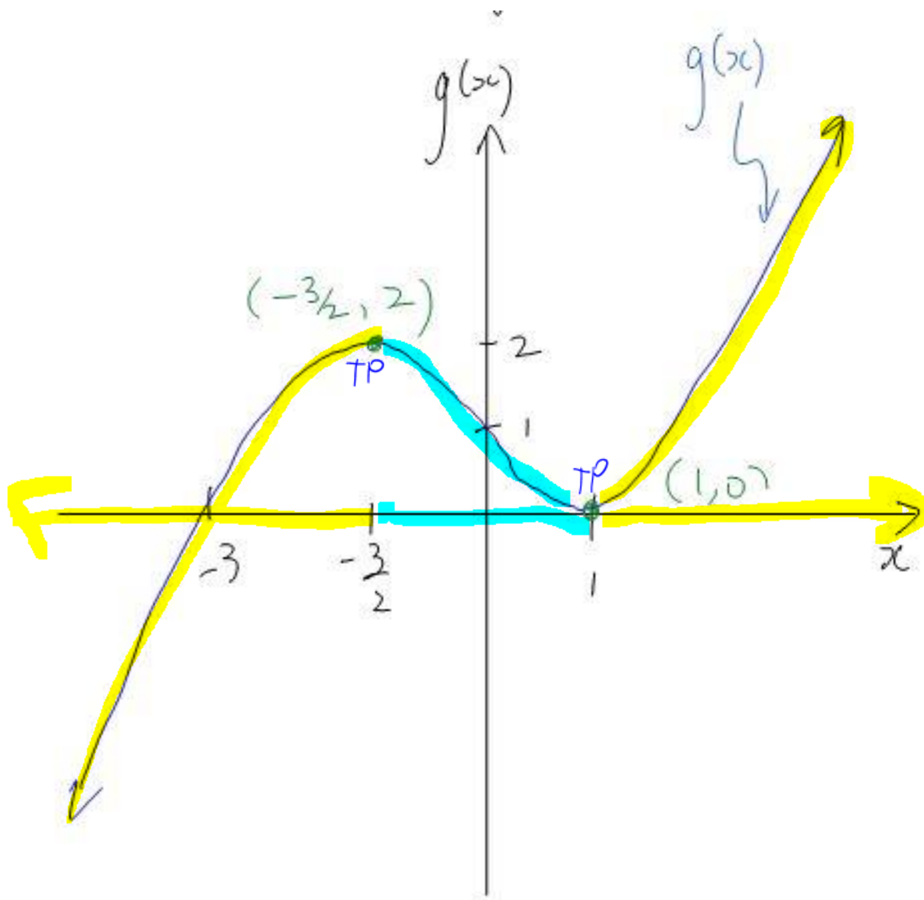


Figure 2.2.2

Observations about $g(x)$:

① $g(x)$ is an odd function and the L.C. is positive
End behaviors are different.

② Two turning points at $x = \frac{-3}{2}$ and $x = 1$

③ Two zeros at $x = -3$ and $x = -1$

④ Local max at $(\frac{-3}{2}, 2)$ and a local min at $(1, 0)$

⑤ Increasing on $(-\infty, \frac{-3}{2}) \cup (1, \infty)$

Decreasing on $(\frac{-3}{2}, 1)$

General Observations about the Behaviour of Polynomial Functions

1) The Domain of all Polynomial Functions is $x \in (-\infty, \infty)$

2) The Range of ODD ORDERED Polynomial Functions is
$f(x) \in (-\infty, \infty)$

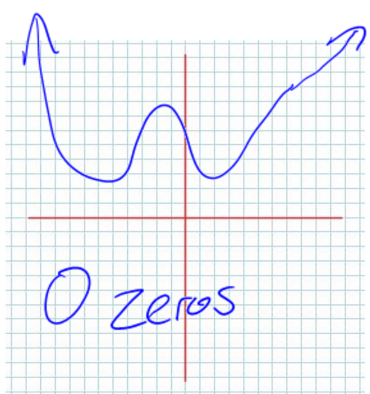
3) The Range of EVEN ORDERED Polynomial Functions

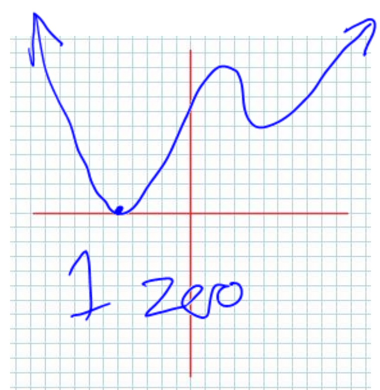
1. The sign of the Leading Coefficient — L.C. > 0, $[\#, \infty)$
2. The y-value from the global max/min — L.C. < 0, $(-\infty, \#]$

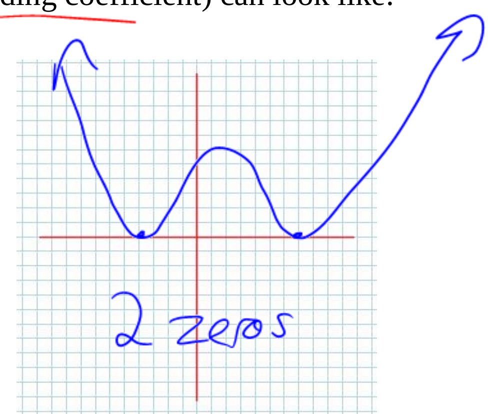
Even Ordered Polynomials

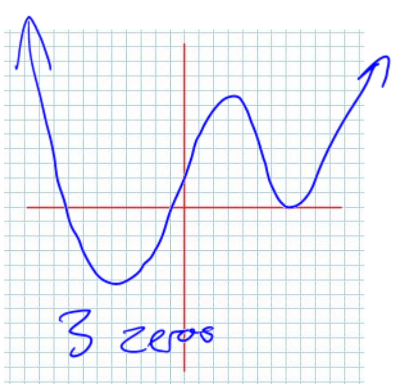
Zeros: A Polynomial Function, $f(x)$, with an even degree of "n" (i.e. $n = 2, 4, 6...$) can have 0 zeros, 1, 2, 3, ... n zeros

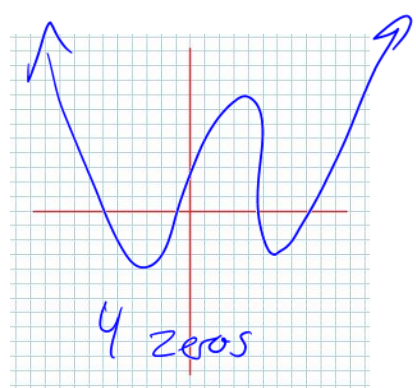
e.g. A degree 4 Polynomial Function (with a positive leading coefficient) can look like:











Turning Points:

The minimum number of turning points for an Even Ordered Polynomial Function is one. It must turn.

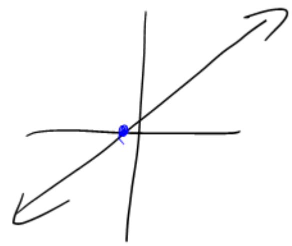
The maximum number of turning points for a Polynomial Function of (even) order *n* is

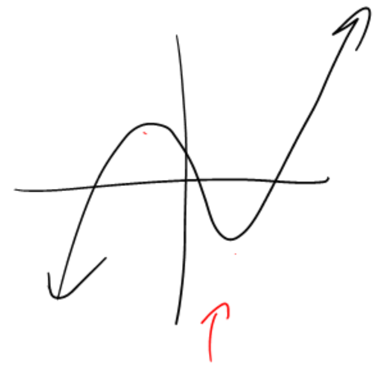
$n-1$

Odd Ordered Polynomials

Zeros: min is one because of opposite end behaviors

max is n.

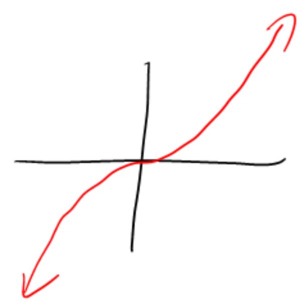




Turning Points:

min # of T.P. is zero.

max # of T.P. $n-1$.



Example 2.2.1 (*#2, for #1b, from Pg. 136*)

Determine the minimum and maximum number of zeros and turning points the given function may have: $g(x)=2x^5-4x^3+10x^2-13x+8$

$g(x)$ is odd

L.C. is positive

Zeros: min $=1$, max $=5$ (n)

T.P.: min $=0$, max $=4$ $(n-1)$

End behaviors $x\to-\infty$ $g(x)\to-\infty$ $x\to\infty$ $g(x)\to\infty$

Example 2.2.2 *(#4d from Pg. 136)*

Describe the end behaviour of the polynomial function using the order and the sign on the leading coefficient for the given function: $f(x) = \boxed{-2x^4} + 5x^3 - 2x^2 + 3x - 1$

↳ negative and even

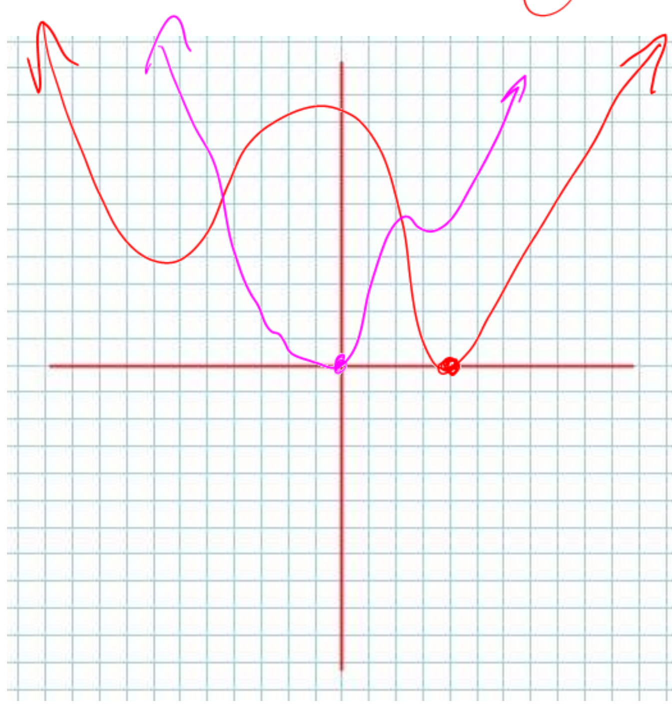
End behaviours $x \to \pm\infty$

$f(x) \to -\infty$

Example 2.2.3 (*#7c from Pg. 137*)

Sketch a graph of a polynomial function that satisfies the given set of conditions:

Degree 4 - positive leading coefficient - 1 zero - 3 turning points.



Symmetry

↳ ignore.

Success Criteria:

- I can differentiate between an even and odd degree polynomial
- I can identify the number of turning points given the degree of a polynomial function
- I can identify the number of zeros given the degree of a polynomial function
- I can determine the symmetry (if present) in polynomial functions

2.3 Zeros of Polynomial Functions

(*Polynomial Functions in Factored Form*)

Today we take a deeper look inside the Box of Mystery, ***carefully examining Zeros of Polynomial Functions***

Learning Goal: We are learning to determine the equation of a polynomial function that describes a particular situation or graph and vice-versa.

We'll begin with an **Algebraic Perspective**:

Consider the polynomial function in factored form:

$$f(x) = (2x-3)(x-1)(x+2)(x+3)$$

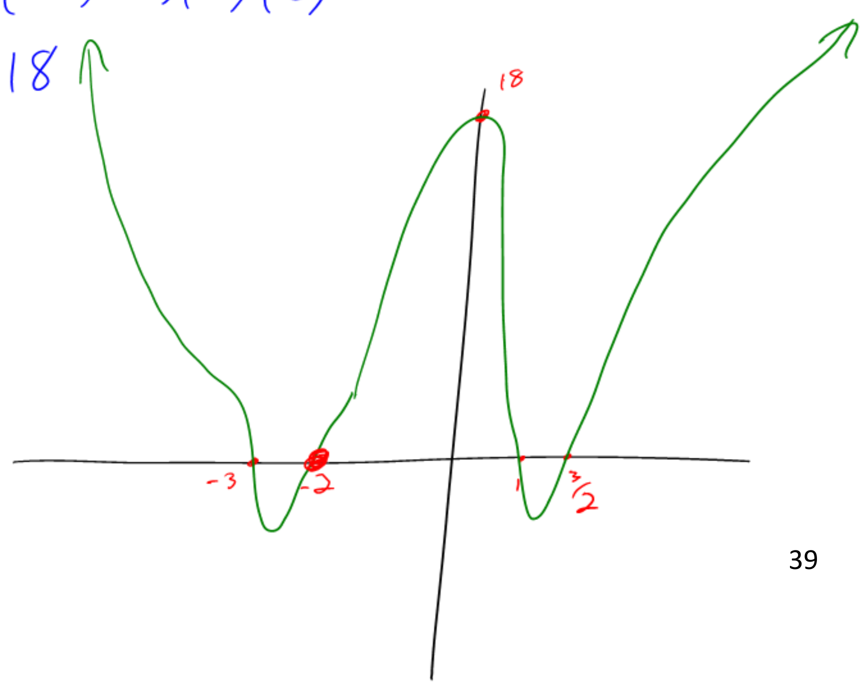
Observations: Leading term is $2x^4$

1. f(x) is even and positive, there $x \to -\infty, f(x) \to \infty$
$x \to +\infty, f(x) \to \infty$

2. The order/degree is 4.

3. f(x) has 4 zeros at $x = \frac{3}{2}, 1, -2, -3$

4. y-int: $f(0) = (-3)(-1)(2)(3)$
$= 18$



$(x-3)^2 = 0$

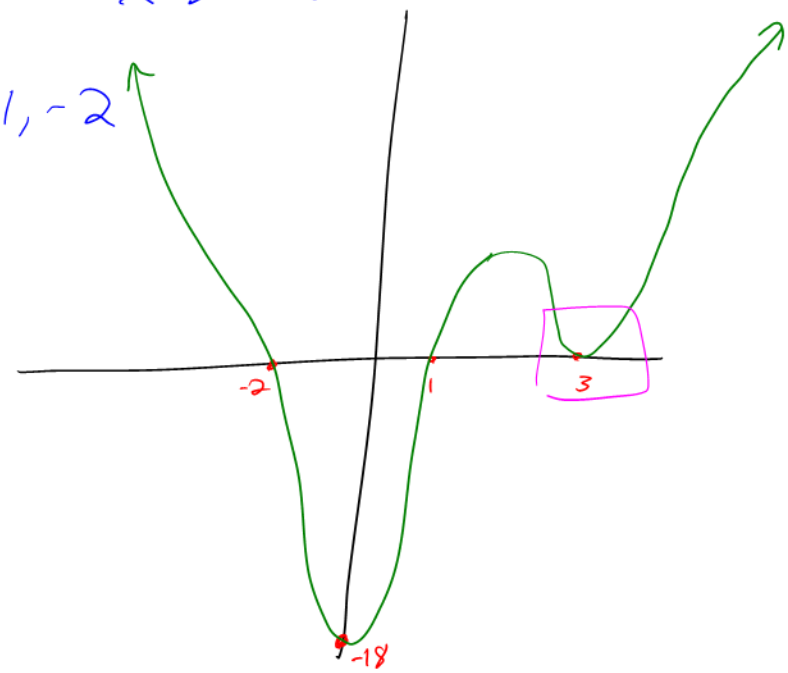
Now, consider the polynomial function $g(x) = \boxed{(x-3)^2}(x-1)(x+2)$

Observations: Leading Term is x^4

1. $g(x)$ is even and positive, $\therefore$ $x \to \pm\infty$, $f(x) \to \infty$

2. $g(x)$ zeros at $x = 3, 1, -2$

3. y-int: $g(0) = (-3)^2(-1)(2)$
$= -18$



Geometric Perspective on Repeated Roots (zeros) of order **2**

Consider the quadratic in factored form: $f(x) = (x-1)^2$

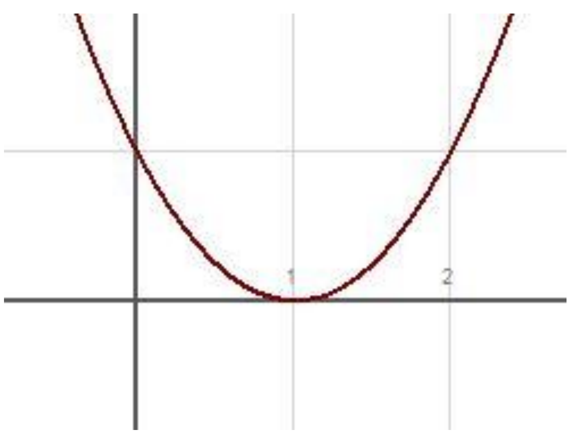


Figure 2.3.1

Consider the polynomial function in factored form: $h(t) = (t+1)^3(2t-5)$

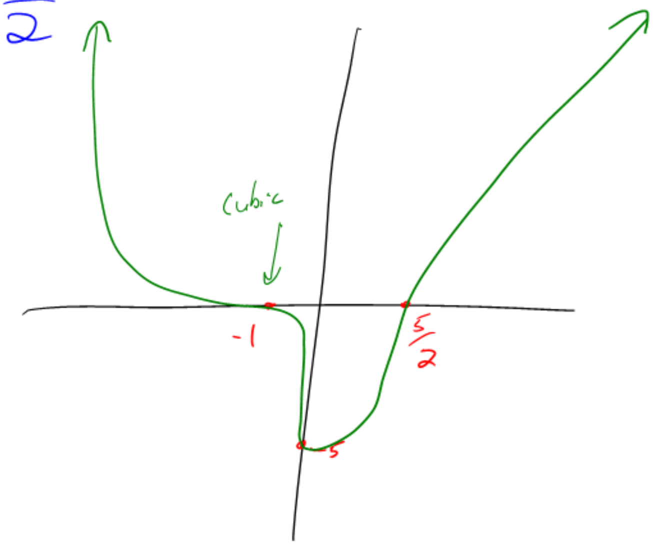
Observations: Leading term: $2t^4$

① $h(t)$ is even, positive $\therefore$ $t \to \pm\infty$, $h(t) \to \infty$

② Zeros at $\boxed{t=-1}$ order 3 and $\frac{5}{2}$

③ y-int: $h(0) = (1)^3(-5)$

$= -5$



Geometric Perspective on Repeated Roots (zeros) of order **3**

Consider the function $f(x) = (x-1)^3$

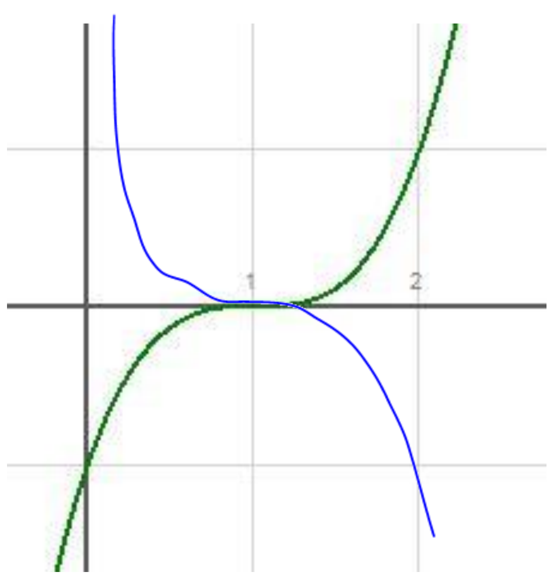
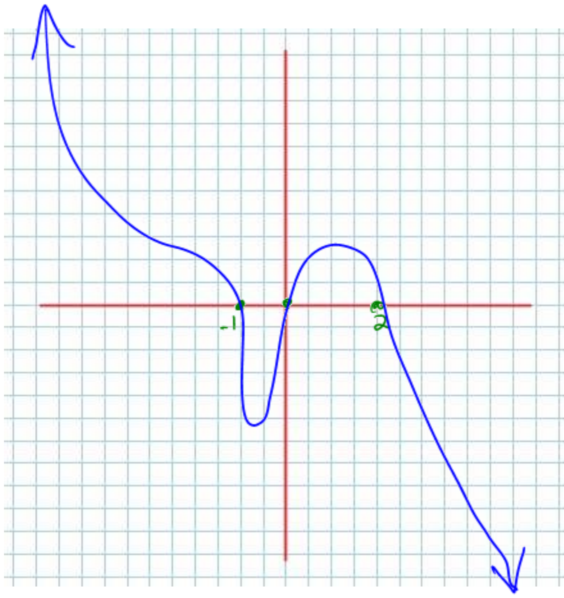


Figure 2.3.2

$f(x)$ is odd and negative

Example 2.3.1

Sketch a (**possible**) graph of $f(x) = -2x(x+1)(x-2)$



L.T. is $-2x^3$

$x \to -\infty,\ f(x) \to \infty$

$x \to \infty,\ f(x) \to -\infty$

order 1 zeros

Zeros: $x = -1, 2, 0$

y-int: $f(0) = -2(0)(0+1)(0-2)$

$= 0(1)(-2)$

$= 0$

Families of Functions

Polynomial functions which share the same order are "broadly related" (e.g. **all** quadratics are in the "order 2 family").

Polynomial Functions which share the same order and zeros are more tightly related.

Polynomial Functions which share the same order, zeros, and end behaviors are like siblings.

$f(x) = -\boxed{2}(x-3)^2(x+1)$

$g(x) = -\boxed{5}(x-3)^2(x+1)$

Example 2.3.2

The family of functions of order 4, with zeros $x = -1, 0, 3, 5$ can be expressed as:

$f(x) = a(x+1)(x+0)(x-3)(x-5)$

↳ this is what distinguishes from family members

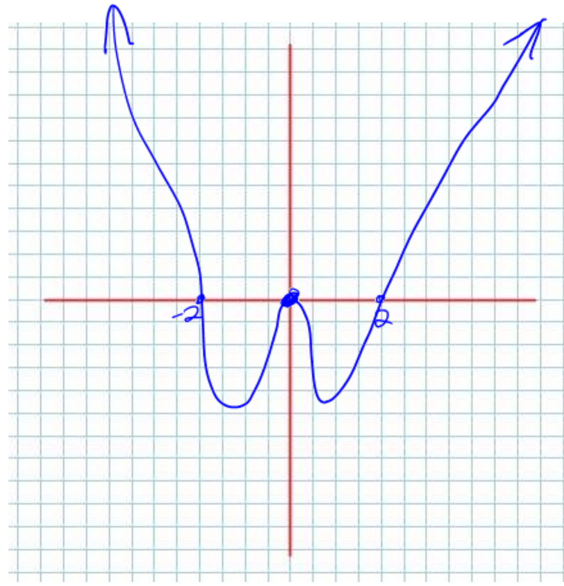
Example 2.3.3

Sketch a graph of $g(x) = 4x^4 - 16x^2$

Leading Term: $4x^4$

$g(x)$ is even, positive

$x \to \pm\infty,\ g(x) \to \infty$



Factor: $g(x) = 4x^2(x^2 - 4)$

$= (4x^2)(x+2)(x-2)$

$x = 0$ order 2 ∴ parabola

$x = -2$

$x = 2$

y-int at 0.

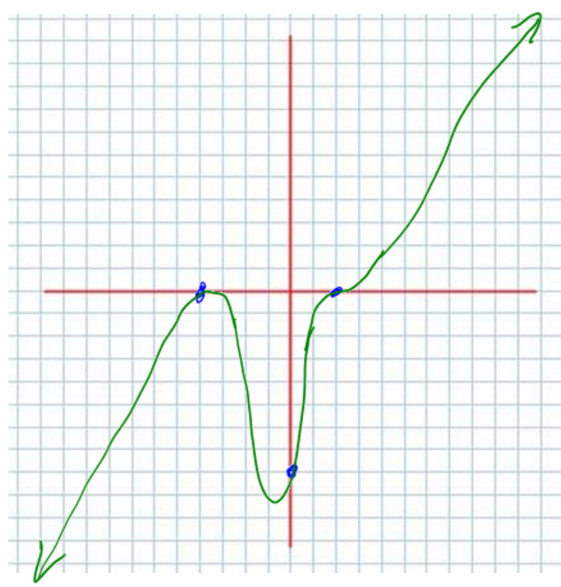
Example 2.3.4

Sketch a (possible) graph of $h(t) = (t-1)^3(t+2)^2$

Leading Term: t^5

Odd, positive, ∴ $t \to -\infty,\ h(t) \to -\infty$

$t \to \infty,\ h(t) \to \infty$



Zeros at $t = 1$ order 3

$t = -2$ order 2

y-int: $h(0) = (-1)^3(2)^2 = -4$

Example 2.3.5

order 4

Determine **the** quartic function, $f(x)$, with zeros at $\underbrace{x=-2,\ 0,\ 1,\ 3}$, if $f(\overset{x}{-1})=-2$.

$f(x) = a(x+2)(x+0)(x-1)(x-3)$

$-2 = a(-1+2)(-1+0)(-1-1)(-1-3)$

$-2 = a(1)(-1)(-2)(-4)$

$-2 = a(-8)$

$\frac{2}{8} = a$

$\frac{1}{4} = a$

$\therefore f(x) = \frac{1}{4}x(x+2)(x-1)(x-3)$

Success Criteria:

- I can determine the equation of a polynomial function in factored form
- I can determine the behaviour of a zero based on the order/exponent of that factor

2.4a Dividing a Polynomial by a Polynomial

(The Hunt for Factors)

Learning Goal: We are learning to divide a polynomial by a polynomial using long division

> *Note: In this course we will almost always be dividing a polynomial by a monomial*

$$\begin{array}{r} 49 \\ 5\overline{)248} \\ -20\downarrow \\ \hline 48 \\ -45 \\ \hline 3 \end{array}$$

$\therefore 248 = 5(49) + 3$

Before embarking, we should consider some "basic" terms (and notation):

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

dividend: the thing you are dividing

divisor: the thing you are dividing by

quotient: the "answer"

remainder: the left overs.

$$\boxed{\text{dividend} = (\text{quotient})(\text{divisor}) + \text{remainder}}$$

The division statement

Note: The Divisor and the Quotient will both be **FACTORS**

IF

The remainder is zero

Example 2.4.1

Use **LONG DIVISION** for the following division problem:

$$\frac{5x^4+3x^3-2x^2+6x-7}{x-2}$$

$$\begin{array}{r} 5x^3+13x^2+24x+54 \\ x-2\,\overline{)\,5x^4+3x^3-2x^2+6x-7} \\ -(5x^4-10x^3) \\ \hline 13x^3-2x^2 \\ -(13x^3-26x^2) \\ \hline 24x^2+6x \\ -(24x^2-48x) \\ \hline 54x-7 \\ -(54x-108) \\ \hline 101 \end{array}$$

Please read Example 1 (Part A) on Pgs. 162 – 163 in your textbook.

$(x)(5x^3) = 5x^4$

$(x)(13x^2) = 13x^3$

$(x)(24x) = 24x^2$

$(x)(54) = 54x$

$\therefore\ 5x^4+3x^3-2x^2+6x-7 = (5x^3+13x^2+24x+54)(x-2)+101$

KEY OBSERVATION:

$(x-2)$ is not a factor

Example 2.4.2

$0x^4$ $0x^2$ must add all terms

Using Long Division, divide $\dfrac{2x^5+3x^3-4x-1}{x-1}$.

$(x)(2x^4)=2x^5$

$$
\begin{array}{r}
2x^4+2x^3+5x^2+5x+1 \\
x-1\,\overline{)\,2x^5+0x^4+3x^3+0x^2-4x-1} \\
-(2x^5-2x^4) \\
\hline
2x^4+3x^3 \\
-(2x^4-2x^3) \\
\hline
5x^3+0x^2 \\
-(5x^3-5x^2) \\
\hline
5x^2-4x \\
-(5x^2-5x) \\
\hline
x-1 \\
-(x-1) \\
\hline
0
\end{array}
$$

$\therefore\ 2x^5+3x^3-4x-1=(x-1)(2x^4+2x^3+5x^2+5x+1)$

KEY OBSERVATION: $(x-1)$ is a factor

Classwork: Pg. 169 #5 **(Yep, that's it for today)**

Success Criteria:

- I can use long division to determine the quotient and remainder of polynomial division
- I can identify a factor of a polynomial if, after long division, there is no remainder

$$
\begin{array}{r}
2x^3 + 4x^2 - 3x + 24 \\
x^2 + 2x - 4 \,\big)\overline{\, 2x^5 + 8x^4 - 3x^3 + 2x^2 - 10x + 8} \\
-(2x^5 + 4x^4 - 8x^3) \\
\hline
4x^4 + 5x^3 + 2x^2 \\
-(4x^4 + 8x^3 - 16x^2) \\
\hline
-3x^3 + 18x^2 - 10x \\
-(-3x^3 - 6x^2 + 12x) \\
\hline
24x^2 - 22x + 8 \\
-(24x^2 + 48x - 96) \\
\hline
-70x + 104
\end{array}
$$

$$(x^2)(2x^3) = 2x^5$$

$$x^2 + 5 \,\big)\overline{\qquad\qquad}$$

$0x$

2.4b Dividing a Polynomial by a Polynomial

(The Hunt for Factors – Part 2)

Learning Goal: We are learning to divide a polynomial by a polynomial using synthetic division

Here we will examine an alternative form of polynomial division called **Synthetic Division**. Don't be fooled! This is not "fake division". You're thinking with the wrong meaning for "synthetic". (Do a search online and see if you can come up with the meaning I am taking!)

In Synthetic Division we concern ourselves with coefficients of the dividend and the zero of the divisor.

we are only using linear divisor : $x-4$ zero $=4$

$2x+5$ zero $=\frac{-5}{2}$

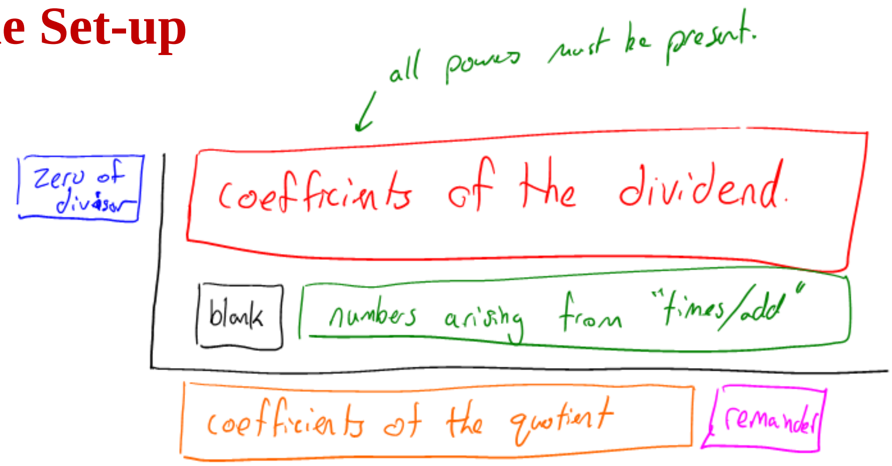
Synthetic Division uses

- only numbers
- 3 Steps: ① Bring down ② times ③ Add

Note:

Only use linear divisors.

The Set-up



Example 2.4.3

Divide using synthetic division:

$$(4x^3-5x^2+2x-1)\div(x-\boxed{2})$$

$$\begin{array}{r|rrrr}
2 & 4 & -5 & 2 & -1 \\
 & \downarrow & 8 & 6 & 16 \\
\hline
 & 4 & 3 & 8 & 15 \\
 & x^2 & x^1 & x^0 & \text{remainder}
\end{array}$$

① Bring Down
② Times
③ Add

$$x^3 \div x = x^2$$

$$\therefore\ 4x^3-5x^2+2x-1=(x-2)(4x^2+3x+8)+15$$

$$3x^4-11x^3-22x^2+15x-25 \div x-5$$

$$\begin{array}{r|rrrrr}
5 & 3 & -11 & -22 & 15 & -25 \\
 & \downarrow & 15 & 20 & -10 & 25 \\
\hline
 & 3 & 4 & -2 & 5 & \boxed{0}
\end{array}$$

$$\therefore\ 3x^4-11x^3-22x^2+15x-25$$
$$=(x-5)(3x^3+4x^2-2x+5)$$

Example 2.4.4

Divide using synthetic division:

$$\frac{4x^4+3x^2-2x+1}{x+1}$$

$$\begin{array}{r|rrrrr}
-1 & 4 & 0 & 3 & -2 & 1 \\
 & \downarrow & -4 & 4 & -7 & 9 \\
\hline
 & 4 & -4 & 7 & -9 & 10
\end{array}$$

$$\therefore\ 4x^4+3x^2-2x+1=(x+1)(4x^3-4x^2+7x-9)+10$$

Example 2.4.5

Divide using your choice of method (*and you choose synthetic division…amen*)

$$(2x^3-9x^2+x+12)\div(2x-3)$$ zero is $\frac{3}{2}$

$$\begin{array}{c|cccc} \frac{3}{2} & 2 & -9 & 1 & 12 \\ & \downarrow & 3 & -9 & -12 \\ \hline & 2 & -6 & -8 & 0 \\ & 1 & -3 & -4 & \end{array}$$

divide all the #'s by the denominator.

$\therefore\ 2x^3-9x^2+x+12=(2x-3)(x^2-3x-4)$

$=(2x-3)(x-4)(x+1)$

Example 2.4.6

Is $3x-1$ a factor of the function $f(x)=6x-x^3+2+3x^4$?

$x=\frac{1}{3}$

$f(x)=3x^4-x^3+0x^2+6x+2$

$$\begin{array}{c|ccccc} \frac{1}{3} & 3 & -1 & 0 & 6 & 2 \\ & \downarrow & 1 & 0 & 0 & 2 \\ \hline & 3 & 0 & 0 & 6 & 4 \end{array}$$

remainder of 4 $\therefore$ $3x-1$ is not a factor.

Example 2.4.7 (OK…this is a lot of examples!)

Consider again (from **Example 2.4.6**) $f(x) = 3x^4 - x^3 + 6x + 2$, and calculate $f\left(\frac{1}{3}\right)$.

$$f\left(\tfrac{1}{3}\right) = 3\left(\frac{1^4}{3^4}\right)^4 - \left(\frac{1^3}{3^3}\right)^3 + 6\left(\frac{1}{3}\right) + 2$$

$$= \frac{\cancel{3}^1}{1}\left(\frac{1}{\cancel{81}_{27}}\right) - \left(\frac{1}{27}\right) + 2 + 2$$

$$= \cancel{\frac{1}{27} - \frac{1}{27}} + 4$$

$$= 4$$ WHOAH! This is the same remainder when dividing by $3x-1$

Example 2.4.8

Consider **Example 2.4.5**. Let $g(x) = 2x^3 - 9x^2 + x + 12$, and calculate $g\left(\frac{3}{2}\right)$.

$$g\left(\tfrac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + \frac{3}{2} + 12$$

$$= \cancel{2}\left(\frac{27}{\cancel{8}_4}\right) - \frac{9}{1}\left(\frac{9}{4}\right) + \frac{3}{2} + \frac{12}{1}$$

$$= \frac{27}{4} - \frac{81}{4} + \frac{6}{4} + \frac{48}{4}$$

$$= 0 !!$$

The Remainder Theorem

Given a polynomial function, $f(x)$, divided by a linear binomial, $x-k$, then the remainder of the division is the value $f(k)$.

Proof of the Remainder Theorem

Consider: $f(x) \div (x - k)$

Then: $f(x) = (x-k)(q(x)) + r$

$f(k) = \cancel{(k-k)(q(k))} + r$

$f(k) = r$ $\square$

Example 2.4.9

Determine the remainder of $\frac{\boxed{5x^4 - 3x^3 - 50}}{x - \boxed{2}}$. $= f(x)$

WAIT!!!! We MUST have a FUNCTION

$f(2) = 5(2)^4 - 3(2)^3 - 50$

$= 80 - 24 - 50$

$= 6$

$$\begin{array}{r|rrrrr} 2 & 5 & -3 & 0 & 0 & -50 \\ & & 10 & 14 & 28 & 56 \\ \hline & 5 & 7 & 14 & 28 & 6 \end{array}$$

Success Criteria:

- I can appreciate that synthetic division is "da bomb"
- I can use synthetic division to determine the quotient and remainder of polynomial division
- I can identify a factor of a polynomial if, after synthetic division, there is no remainder (The Remainder Theorem)

2.5 The Factor Theorem

(Factors have been FOUND)

Learning Goal: We are learning the connections between a polynomial function and its remainder when divided by a binomial

The Factor Theorem

Given a polynomial function, $f(x)$, then $x-a$ is a factor of $f(x)$ if and only if $f(a)=0$.

$(x-5)(x+2)$

-10

Example 2.5.1

Use the Factor Theorem to factor x^3+2x^2-5x-6.

WAIT!!!! We need a FUNCTION

$f(x)=(x-a)(x-b)(x-c)$

$(a)(b)(c)=-6$

we need factors of 6.

$f(x)=x^3+2x^2-5x-6$

Test the possible factors of 6

$\pm1, \pm2, \pm3, \pm6$

Test $x=1$ $(x-1)$

$f(1)=1^3+2(1)^2-5(1)-6$

$=1+2-5-6$

$\neq 0$ $\therefore$ not a factor

Test $x=-1$ $\boxed{(x+1)}$

$f(-1)=(-1)^3+2(-1)^2-5(-1)-6$

$=-1+2+5-6$

$=0$ $\therefore$ a factor, so divide with it.

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1x^2 & 1x & -6 & 0 \end{array}$$

A M

$\therefore x^3+2x^2-5x-6=(x+1)(x^2+x-6)$

$=(x+1)(x+3)(x-2)$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24, \pm 48$

Example 2.5.2

Factor **fully** $x^4 - x^3 - 16x^2 + 4x \boxed{+48}$

Test $x = 1$

$f(1) = 1^4 - 1^3 - 16(1)^2 + 4(1) + 48$

$= 1 - 1 - 16 + 4 + 48$

$\neq 0$

Test $x = -2$ $(x+2)$

$f(-2) = (-2)^4 - (-2)^3 - 16(-2)^2 + 4(-2) + 48$

$= 16 + 8 - 64 - 8 + 48$

$=$ ☺

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -16 & +4 & +48 \\ & & -2 & 6 & 20 & -48 \\ \hline & 1 & -3 & -10 & 24 & 0 \end{array}$$

$\therefore (x+2)(x^3 - 3x^2 - 10x + \underline{24})$

Test $x = 2$

$g(2) = (2)^3 - 3(2)^2 - 10(2) + 24$

$= 8 - 12 - 20 + 24$

$=$ ☺

$$\begin{array}{r|rrrr} 2 & 1 & -3 & -10 & 24 \\ & & 2 & -2 & -24 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

$\therefore x^4 - x^3 - 16x^2 + 4x + 48 = (x+2)(x-2)(x^2 - x - 12)$

$= (x+2)(x-2)(x-4)(x+3)$

Example 2.5.3 (*Pg 177 #6c in your text***)**

Factor fully $x^4+8x^3+4x^2-48x$

$$= x(\underbrace{x^3+8x^2+4x-48}_{g(x)})$$

Try $x=2$: $g(2) = 2^3 + 8(2)^2 + 4(2) - 48$

$= 8 + 32 + 8 - 48$

$= 0$

$$\begin{array}{r|rrrr} 2 & 1 & 8 & 4 & -48 \\ & & 2 & 20 & 48 \\ \hline & 1 & 10 & 24 & 0 \end{array}$$

$\therefore x^4+8x^3+4x^2-48x = x(x-2)(x^2+10x+24)$

$= x(x-2)(x+4)(x+6)$

Example 2.5.4 (*Pg 177 #10*)

$f(1)=10$

When ax^3-x^2+2x+b is divided by $x-1$ the remainder is 10. When it is divided by $x-2$ the remainder is 51. Find *a* and *b*.

$f(2)=51$

This problem is very instructive.

$f(x) = ax^3 - x^2 + 2x + b$

$10 = a(1)^3 - (1)^2 + 2(1) + b$

$10 = a - 1 + 2 + b$

$9 = a + b$

$9 = a + 51 - 8a$

$-42 = -7a \quad \therefore a = 6$ and $b = 3$

$51 = a(2)^3 - (2)^2 + 2(2) + b$

$51 = 8a - 4 + 4 + b$

$51 = 8a + b$

$\boxed{51 - 8a} = b$

$$\begin{array}{rl} a + \cancel{b} &= 9 \\ -(8a + \cancel{b} &= 51) \\ \hline -7a &= -42 \\ a &= 6 \end{array}$$

Success Criteria:

- I can use test values to find the factors of a polynomial function
- I can factor a polynomial of degree three or greater by using the factor theorem
- I can recognize when a polynomial function is not factorable

2.6 Factoring Sums and Differences of Cubes

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Knowing how to factor a sum or difference of cubes is a simple matter of remembering patterns.

Learning Goal: We are learning to factor a sum or difference of cubes.

Example 2.6.1 (*Recalling the pattern for factoring a Difference of Squares*)

Factor $4x^2 - 25$

$= (2x - 5)(2x + 5)$

Note: Sums of Squares DO NOT factor!!

e.g. Simplify $x^2 + 4$

Differences of Cubes $8x^3 - 27$

Same
Opposite
Always
Positive

Pattern

$(cube_1 - cube_2) = (cuberoot_1 - cuberoot_2)(cuberoot_1^2 + cuberoot_1 \times cuberoot_2 + cuberoot_2^2)$

$8x^3 - 27 \quad (2x - 3)(4x^2 + 6x + 9$

TWO POSITIVES and ONE NEGATIVE

Sums of Cubes (These DO factor!!)

$8x^3 + 27$

Pattern

$(cube_1 + cube_2) = (cuberoot_1 + cuberoot_2)(cuberoot_1^2 - cuberoot_1 \times cuberoot_2 + cuberoot_2^2)$

$(2x + 3)(4x^2 - 6x + 9)$

Same Opposite Always Positive

Example 2.6.2

Factor $x^3 - 8$ $= (x-2)(x^2+2x+4)$

Example 2.6.3

Factor $27x^3 + 125y^3$ $= (3x+5y)(9x^2-15xy+25y^2)$

Example 2.6.4

Factor $1-64z^3$ $= (1-4z)(1+4z+16z^2)$

Example 2.6.5

Factor $1000x^3+27$ $= (10x+3)(100x^2-30x+9)$

Example 2.6.6

Factor x^6-729 $= (x^2-9)(x^4+9x^2+81)$

$(x^2)^3 = x^6$ $= (x-3)(x+3)(x^4+9x^2+81)$

Success Criteria:

- I can use patterns to factor a sum of cubes
- I can use patterns to factor a difference of cubes